



# Scheduling jobs with an exponential sum-of-actual-processing-time-based learning effect

Ji-Bo Wang<sup>a,d,f,\*</sup>, Lin-Hui Sun<sup>b,d,e,f</sup>, Lin-Yan Sun<sup>c,d,e,f</sup>

<sup>a</sup> School of Science, Shenyang Aerospace University, Shenyang 110136, China

<sup>b</sup> School of Economics and Management, Xi'an University of Technology, Xi'an 710054, China

<sup>c</sup> School of Management, Xi'an Jiaotong University, Xi'an 710049, China

<sup>d</sup> The State Key Laboratory for Manufacturing Systems Engineering, Xi'an 710054, China

<sup>e</sup> The Key Laboratory of the Ministry of Education for Process Control and Efficiency Engineering, Xi'an 710049, China

<sup>f</sup> Knowledge Management & Innovation Research Centre of Xi'an Jiaotong University, Xi'an 710049, China

## ARTICLE INFO

### Article history:

Received 27 March 2009

Received in revised form 1 September 2010

Accepted 1 September 2010

### Keywords:

Scheduling

Single machine

Learning effect

## ABSTRACT

In this paper we consider single-machine scheduling problems with an exponential sum-of-actual-processing-time-based learning effect. By the exponential sum-of-actual-processing-time-based learning effect, we mean that the processing time of a job is defined by an exponential function of the sum of the actual processing times of the already processed jobs. For the proposed learning model, we show that under certain conditions, the makespan minimization problem, the sum of the  $\theta$ th ( $\theta > 0$ ) powers of completion times minimization problem, and some special cases of the total weighted completion time minimization problem and the maximum lateness minimization problem all remain polynomially solvable.

© 2010 Elsevier Ltd. All rights reserved.

## 1. Introduction

In classical scheduling theory, it is assumed that the processing time of a job is a constant. However, in many realistic problems of operations management, both machines and workers can improve their performance by repeating the production operations. Therefore, the actual processing time of a job is shorter if it is scheduled later in a sequence. This phenomenon is known as the “learning effect” in the literature [1]. An extensive survey of different scheduling models and problems with learning effects can be found in [2]. More recent papers which have considered scheduling jobs with learning effects include Wang et al. [3], Mosheiov [4], Gordon et al. [5], Gordon and Strusevich [6], Wang et al. [7] and Wang [8]. Wang et al. [3] considered single-machine scheduling problems with time-dependent learning effect. They proved that the weighted shortest processing time (WSPT) rule, the earliest due date (EDD) rule and the modified Moore–Hodgson algorithm can, under certain conditions, construct the optimal schedule for the problem of minimizing the following three objectives: the total weighted completion time, the maximum lateness and the number of tardy jobs, respectively. They also gave an error estimation for each of these rules for the general cases. Mosheiov [4] considered the problem of minimizing the total absolute deviation of job completion times (TADC). He showed that with both extensions (simultaneously), i.e., (i) position-dependent processing times and (ii) parallel identical machines, the problem of minimizing the sum of the TADC values on all the machines remains polynomially solvable. Gordon et al. [5] considered single-machine scheduling problems where the processing time of a job depends either on its position in a processing sequence or on its start time. They showed that in many situations the makespan minimization problem and the sum of the (weighted) completion times minimization

\* Corresponding author at: School of Science, Shenyang Aerospace University, Shenyang 110136, China.

E-mail address: [wangjibo75@yahoo.com.cn](mailto:wangjibo75@yahoo.com.cn) (J.-B. Wang).

problem are polynomially solvable under series-parallel precedence constraints. Gordon and Strusevich [6] considered single-machine due date assignment scheduling with positionally dependent processing times. They gave polynomial-time dynamic programming algorithms for two popular due date assignment methods: CON and SLK. Wang et al. [7] considered the single-machine scheduling problem with exponential time-dependent learning effect and past-sequence-dependent (p-s-d) setup times. They showed that the makespan minimization problem, the total completion time minimization problem and the sum of the quadratic job completion times minimization problem can be solved by using the smallest (normal) processing time first (SPT) rule, respectively.

The concept of learning effects has been extensively studied in the literature [2]. However, to the best of our knowledge, apart from the recent papers of Yang and Kuo [9] and Wang [8], the scheduling problem with a sum-of-actual-processing-time-based learning effect has not been investigated. The phenomena of a sum-of-actual-processing-time-based learning effect can be found in real-life situations. In many situations, the operating processes of a job differ, for example, car repair/maintenance and patient diagnosis/treatment. The conditions of cars or patients are different. Hence, there are no identical repetitions of operating processes in the job. Nevertheless, there still exists a certain learning effect after operating the job. In such situations, the learning effect is due to the experience of operating jobs. This also implies that the learning effect depends on the total actual processing time of jobs. Therefore, the actual processing time of a job is affected by the total actual processing time of the previous jobs' schedules [9]. Yang and Kuo [9] considered single-machine scheduling with an actual time-dependent learning effect. They showed that the problem remains polynomially solvable for the following three objectives: the makespan, the total completion time and the sum of the  $k$ th powers of completion times. They also proved that the sum of the weighted completion times minimization problem remains polynomially solvable if jobs have reasonable weights. Wang [8] considered single-machine scheduling with a sum-of-actual-processing-time-based learning effect. He showed that the makespan minimization problem, the total completion time minimization problem, and the total completion time square minimization problem can be solved by using the SPT rule. In this paper we consider a new model different from those of Yang and Kuo [9] and Wang [8] where the learning effect is expressed as an exponential function of the sum of the actual processing time of the already processed jobs. This model is motivated by the ideas of Yang and Kuo [9], Wang et al. [7] and Wang [8]. The phenomena of an exponential sum-of-actual-processing-time-based learning effect can be found in real-life situations. For example, for memory chip processes, the conditions of memory chips are different. In such processes, the exponential sum-of-actual-processing-time-based learning effect is due to the experience of operating jobs. Under the proposed model, the actual processing time of a job is defined by an exponential function of the sum of actual processing times of the jobs already processed.

The remaining part of this paper is organized as follows. In Section 2 we formulate the model. In Section 3 we consider several single-machine scheduling problems. The last section presents the conclusions.

## 2. Problem formulation

The focus of this section is on studying a new learning effect model in single-machine scheduling. The model is described as follows. There are given a single machine and  $n$  independent and non-preemptive jobs that are immediately available for processing. The machine can handle one job at a time and preemption is not allowed. Let  $p_j$  be the normal processing time of job  $J_j$ ,  $p_{[k]}^A$  be the normal (actual) processing time of a job if it is scheduled in the  $k$ th position in a sequence. Associated with each job  $J_j$ , there is a weight  $w_j$  and a due date  $d_j$ . Let  $p_{jr}^A$  be the actual processing time of job  $J_j$  if it is scheduled in position  $r$  in a sequence. In this paper, we consider a new learning effect model, i.e.,

$$p_{jr}^A = p_j \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right), \quad r, j = 1, 2, \dots, n, \quad (1)$$

where  $\alpha \geq 0$ ,  $\beta \geq 0$  and  $0 < a \leq 1$  are parameters obtained empirically,  $\alpha + \beta = 1$  and  $\sum_{i=1}^0 p_{[i]}^A := 0$ .

For a given schedule  $\pi = (J_{[1]}, J_{[2]}, \dots, J_{[n]})$ , where  $J_{[j]}$  denotes the job that occupies the  $j$ th position in  $\pi$ ,  $C_j = C_j(\pi)$  represents the completion time of job  $J_j$ . Let  $C_{\max} = \max\{C_j | j = 1, 2, \dots, n\}$ ,  $\sum C_j^\theta$ ,  $\sum w_j C_j$  and  $L_{\max} = \max\{C_j - d_j | j = 1, 2, \dots, n\}$  represent the makespan, the sum of the  $\theta$ th ( $\theta > 0$ ) powers of completion times, the total weighted completion time and the maximum lateness of a given permutation, respectively. In the remaining part of the paper, all the problems considered will be denoted using the three-field notation scheme  $\alpha|\beta|\gamma$  introduced by Graham et al. [10].

## 3. Single-machine scheduling problems

### 3.1. The makespan minimization problem

In the classical single-machine makespan minimization scheduling problem, the makespan value is sequence independent. However, this may be different when the learning effect is considered. Wang et al. [7] showed that the SPT rule is optimal for the single-machine makespan minimization scheduling problem with an exponential time-dependent learning effect. But the following example shows that the SPT order does not yield an optimal schedule under the proposed learning model.

**Counter-Example 1.**  $n = 3, p_1 = 1, p_2 = 2, p_3 = 20, \alpha = \beta = a = 0.5$ . The SPT sequence is  $[J_1, J_2, J_3]$ ,  $C_{\max} = 1 + 2 * (0.5 * 0.5^1 + 0.5) + 20 * (0.5 * 0.5^{1+2*(0.5*0.5^1+0.5)} + 0.5) = 14.2678$ . Obviously, the optimal sequence is  $[J_2, J_1, J_3]$ ,  $C_{\max}^* = 2 + 1 * (0.5 * 0.5^2 + 0.5) + 20 * (0.5 * 0.5^{2+1*(0.5*0.5^2+0.5)} + 0.5) = 14.2460$ .

Although the SPT order does not provide an optimal schedule under the proposed learning effect model, it still gives an optimal solution if the processing times of jobs satisfy certain conditions. First, we give some lemmas; they are useful for the following theorems.

**Lemma 1.**  $(\delta a^{\lambda x} - 1) - \lambda(\delta a^x - 1) \geq 0$  if  $\lambda \geq 1, 0 < a \leq 1, 0 \leq \delta \leq 1$ , and  $x \geq 0$ .

**Proof.** See the proof of Wang et al. [7].  $\square$

**Lemma 2.**  $a^{x+y} - a^x \geq y \ln a$  if  $x \geq 0, y \geq 0$  and  $0 < a \leq 1$ .

**Proof.** Let  $f(z) = a^z$  ( $z > 0$ ). Then we have  $f'(z) = a^z \ln a$  and  $f''(z) = a^z \ln^2 a \geq 0$ . Hence,  $f'(z)$  is increasing on  $z > 0$  and  $0 < a \leq 1$  for  $f''(z) \geq 0$ . According to the mean value theorem, for any  $z > 0$  and  $z_0 > 0$ , there exists a point  $\zeta$  between  $z$  and  $z_0$  such that  $f(z) - f(z_0) = f'(\zeta)(z - z_0)$ . Let  $z = x + y$  and  $z_0 = x$ . Then we have  $a^{x+y} - a^x = ya^\zeta \ln a \geq ya^x \ln a \geq y \ln a$ .  $\square$

**Theorem 1.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)|C_{\max}$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$ , then an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $p_j$  (the SPT rule).

**Proof.** Let  $\pi$  and  $\pi'$  be two job schedules where the difference between  $\pi$  and  $\pi'$  is a pairwise interchange of two adjacent jobs  $J_j$  and  $J_k$ , that is,  $\pi = [S_1, J_j, J_k, S_2]$ ,  $\pi' = [S_1, J_k, J_j, S_2]$ , where  $S_1$  and  $S_2$  are partial sequences. Furthermore, we assume that there are  $r - 1$  jobs in  $S_1$ . Thus,  $J_j$  and  $J_k$  are the  $r$ th and the  $(r + 1)$ th jobs, respectively, in  $\pi$  and with  $p_j \leq p_k$ . Likewise,  $J_k$  and  $J_j$  are scheduled in the  $r$ th and the  $(r + 1)$ th positions in  $\pi'$ . To further simplify the notation, let  $B$  denote the completion time of the last job in  $S_1$  and  $J_h$  be the first job in  $S_2$ . To show that  $\pi$  dominates  $\pi'$ , it suffices to show that  $C_k(\pi) \leq C_j(\pi')$  and  $C_u(\pi) \leq C_u(\pi')$  for any  $J_u$  in  $S_2$ .

Under  $\pi$ , the completion times of jobs  $J_j$  and  $J_k$  are

$$C_j(\pi) = B + p_j \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right). \quad (2)$$

$$C_k(\pi) = B + p_j \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right) + p_k \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_j} \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right) + \beta \right). \quad (3)$$

However, under  $\pi'$ , they are

$$C_k(\pi') = B + p_k \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right) \quad (4)$$

and

$$C_j(\pi') = B + p_k \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right) + p_j \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_k} \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right) + \beta \right). \quad (5)$$

On the basis of Eqs. (3) and (5), we have

$$C_j(\pi') - C_k(\pi) = \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} \left[ p_j \left( a^{p_k \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right)} - 1 \right) - p_k \left( a^{p_j \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta \right)} - 1 \right) \right]. \quad (6)$$

Substituting  $\lambda = p_k/p_j$  and  $x = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)$  into (6), we derive from Lemma 1 that

$$C_j(\pi') - C_k(\pi) = \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} p_j [(\delta a^{\lambda x} - 1) - \lambda(a^x - 1)] \geq 0$$

since  $\lambda = p_k/p_j \geq 1, 0 < a \leq 1$  and  $x \geq 0$ ; hence  $C_j(\pi') - C_k(\pi) \geq 0$ .

Note that

$$C_h(\pi') = C_j(\pi') + p_h(\alpha a^{C_j(\pi')} + \beta) \quad (7)$$

and

$$C_h(\pi) = C_k(\pi) + p_h(\alpha a^{C_k(\pi)} + \beta). \quad (8)$$

Taking the difference between Eqs. (7) and (8), we have

$$\begin{aligned}
C_h(\pi') - C_h(\pi) &= C_j(\pi') - C_k(\pi) + \alpha p_h [a^{C_j(\pi')} - a^{C_k(\pi)}] \\
&= C_j(\pi') - C_k(\pi) + \alpha p_h [a^{C_k(\pi) + C_j(\pi') - C_k(\pi)} - a^{C_k(\pi)}] \\
&\geq C_j(\pi') - C_k(\pi) + \alpha p_h \ln a (C_j(\pi') - C_k(\pi)) \quad (\text{from Lemma 2}) \\
&= (C_j(\pi') - C_k(\pi))(1 + \alpha p_h \ln a) \\
&\geq 0 \quad \left( \text{due to } p_j \leq \frac{-1}{\alpha \ln a} \right).
\end{aligned}$$

Thus  $C_h(\pi') \geq C_h(\pi)$ . In other words, we have shown that the first job  $J_h$  in  $S_2$ , which starts earlier in  $\pi$  than in  $\pi'$ , completes earlier in  $\pi$ . Similarly, we have  $C_u(\pi') \geq C_u(\pi)$  for any  $J_u$  in  $S_2$ . This completes the proof.  $\square$

### 3.2. The sum of the $\theta$ th powers of completion times minimization problem

Townsend [11] considered a single-machine scheduling problem with quadratic objective. He showed that the problem  $1 \parallel \sum C_j^2$  can be solved optimally by sequencing jobs in non-decreasing order of their normal processing times (the SPT rule). By a proof similar to that for Theorem 1, we can show that Townsend's solution still holds for the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)| \sum C_j^\theta$  under certain conditions.

**Theorem 2.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)| \sum C_j^\theta$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$ , then an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $p_j$  (the SPT rule).

**Proof.** This is similar to the proof of Theorem 1 except that  $C_j(\pi') \geq C_k(\pi)$ ,  $C_k(\pi') \geq C_j(\pi)$ , and  $C_u(\pi') \geq C_u(\pi)$  for any  $J_u$  in  $S_2$ ; hence

$$\sum C_j^\theta(\pi') \geq \sum C_j^\theta(\pi).$$

This completes the proof.  $\square$

**Corollary 1.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)| \sum C_j$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$ , then an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $p_j$  (the SPT rule).

### 3.3. The total weighted completion time minimization problem

**Theorem 3.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)| \sum w_j C_j$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$  and the jobs have reasonable weights, i.e.,  $p_j < p_k$  implies  $w_j \geq w_k$  for all the jobs  $J_j$  and  $J_k$ , an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $p_j/w_j$  (the WSPT rule).

**Proof.** Here, we still use the same notation as in the proof of Theorem 1. Suppose that  $p_j/w_j \leq p_k/w_k$ , which implies  $p_j \leq p_k$  due to the weights of jobs being reasonable. In order to prove that the WSPT schedule is optimal for the problem, it is sufficient to show that  $w_j C_j(\pi) + w_k C_k(\pi) \leq w_k C_k(\pi') + w_j C_j(\pi')$  since  $C_u(\pi) \leq C_u(\pi')$  for any  $J_u$  in  $S_2$  by the proof of Theorem 1. From Eqs. (2)–(5), we have

$$\begin{aligned}
w_j C_j(\pi) + w_k C_k(\pi) - w_k C_k(\pi') - w_j C_j(\pi') &= \beta(w_k p_j - w_j p_k) + (w_j + w_k)(p_j - p_k) \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} \\
&\quad + w_k p_k \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)} \right) - w_j p_j \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_k(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)} \right).
\end{aligned}$$

Let  $\lambda_1 = \frac{w_j}{w_j + w_k}$ ,  $\lambda_2 = \frac{w_k}{w_j + w_k}$ ,  $t = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)$  and  $\lambda = \frac{p_k}{p_j}$ . Then  $(w_j + w_k)(p_j - p_k) \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + w_k p_k (\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)}) - w_j p_j (\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_k(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)})$  can be rewritten as

$$\begin{aligned}
&(w_j + w_k)(p_j - p_k) \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + w_k p_k \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)} \right) - w_j p_j \left( \alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A + p_k(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)} \right) \\
&= \alpha(w_j + w_k) a^{\sum_{i=1}^{r-1} p_{[i]}^A} p_j [(1 - \lambda_1 a^{\lambda t}) - \lambda(1 - \lambda_2 a^t)] \\
&\leq \alpha(w_j + w_k) a^{\sum_{i=1}^{r-1} p_{[i]}^A} p_j [(1 - \lambda_2 a^{\lambda t}) - \lambda(1 - \lambda_2 a^t)] \\
&\leq 0 \quad (\text{from Lemma 1})
\end{aligned}$$

since  $p_j/w_j \leq p_k/w_k$ ,  $p_j \leq p_k$ ,  $w_j \geq w_k$  and  $\lambda_1 \geq \lambda_2$ . Hence,  $w_j C_j(\pi) + w_k C_k(\pi) \leq w_k C_k(\pi') + w_j C_j(\pi')$ . Thus, repeating this interchange argument for all the jobs not sequenced according to the WSPT rule completes the proof of [Theorem 3](#).  $\square$

Using a method similar to that of [Theorem 3](#), the following corollaries can be easily obtained.

**Corollary 2.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)$ ,  $p_j = p|\sum w_j C_j$ , an optimal schedule can be obtained by sequencing the jobs in non-increasing order of  $w_j$ .

**Corollary 3.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)$ ,  $w_j = kp_j|\sum w_j C_j$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$ , then an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $p_j$  (the SPT rule).

### 3.4. The maximum lateness minimization problem

**Theorem 4.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)|L_{\max}$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$  and if the jobs have reasonable conditions, i.e.,  $p_i < p_j$  implies  $d_i \leq d_j$  for all the jobs  $J_i$  and  $J_j$ , an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $d_j$  (the EDD rule).

**Proof.** Consider an optimal schedule  $\pi$  that does not follow the EDD rule. In this schedule there must be at least two adjacent jobs, say  $J_j$  and  $J_k$  in the  $r$ th and  $(r+1)$ th positions of  $\pi$ , respectively, such that  $d_j > d_k$ , which implies  $p_j \geq p_k$ . Schedule  $\pi'$  is obtained from schedule  $\pi$  by interchanging jobs in the  $r$ th and in the  $(r+1)$ th positions of  $\pi$ . Under  $\pi$ , the latenesses of the jobs are

$$\begin{aligned} L_j(\pi) &= C_j(\pi) - d_j, \\ L_k(\pi) &= C_k(\pi) - d_k, \end{aligned}$$

whereas under  $\pi'$ , they are

$$\begin{aligned} L_k(\pi') &= C_k(\pi') - d_k, \\ L_j(\pi') &= C_j(\pi') - d_j. \end{aligned}$$

Since  $d_j > d_k$  and  $p_j \geq p_k$ , we have  $C_j(\pi') \leq C_k(\pi)$  and  $C_k(\pi') \leq C_j(\pi)$  (like in the proof of [Theorem 1](#)), so it is easily verified that  $L_j(\pi) < L_k(\pi)$ ,  $L_k(\pi') < L_k(\pi)$ ,  $L_j(\pi') < L_k(\pi)$  and

$$\max\{L_j(\pi'), L_k(\pi')\} < \max\{L_j(\pi), L_k(\pi)\}.$$

Hence, interchanging the positions of jobs  $J_j$  and  $J_k$  decreases the value of  $L_{\max}$ . This is a contradiction.  $\square$

Using a method similar to that for [Theorem 4](#), the following corollaries can be easily obtained.

**Corollary 4.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)$ ,  $p_j = p|L_{\max}$ , an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $d_j$  (the EDD rule).

**Corollary 5.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)$ ,  $d_j = d|L_{\max}$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$ , then an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $p_j$  (the SPT rule).

**Corollary 6.** For the problem  $1|p_{jr}^A = p_j(\alpha a^{\sum_{i=1}^{r-1} p_{[i]}^A} + \beta)$ ,  $d_j = kp_j|L_{\max}$ , if  $p_j \leq \frac{-1}{\alpha \ln a}$ , then an optimal schedule can be obtained by sequencing the jobs in non-decreasing order of  $d_j$  (the EDD rule).

## 4. Conclusions

In this paper we considered the single-machine scheduling problems with an exponential sum-of-actual-processing-time-based learning effect. The exponential sum-of-actual-processing-time-based learning effect of a job is assumed to be an exponential function of the sum of the actual processing times of the already processed jobs. Under the proposed learning model, we showed that under certain conditions, the makespan minimization problem, the sum of the  $\theta$ th powers of completion times minimization problem, and some special cases of the total weighted completion time minimization problem and the maximum lateness minimization problem can be solved in polynomial time.

## Acknowledgements

We are grateful to two anonymous referees for their helpful comments on an earlier version of this paper. This research was supported by the National Natural Science Foundation of China under grant number 11001181.

## References

- [1] A.B. Badiru, Computational survey of univariate and multivariate learning curve models, IEEE Transactions on Engineering Management 39 (1992) 176–188.
- [2] D. Biskup, A state-of-the-art review on scheduling with learning effects, European Journal of Operational Research 188 (2008) 315–329.

- [3] J.-B. Wang, C.T. Ng, T.C.E. Cheng, L.L. Liu, Single-machine scheduling with a time-dependent learning effect, *International Journal of Production Economics* 111 (2008) 802–811.
- [4] G. Mosheiov, Minimizing total absolute deviation of job completion times: extensions to position-dependent processing times and parallel identical machines, *Journal of the Operational Research Society* 59 (2008) 1422–1424.
- [5] V.S. Gordon, C.N. Potts, V.A. Strusevich, J.D. Whitehead, Single machine scheduling models with deterioration and learning: handling precedence constraints via priority generation, *Journal of Scheduling* 11 (2008) 357–370.
- [6] V.S. Gordon, V.A. Strusevich, Single machine scheduling and due date assignment with positionally dependent processing times, *European Journal of Operational Research* 198 (2009) 57–62.
- [7] J.-B. Wang, D. Wang, L.-Y. Wang, L. Lin, N. Yin, W.-W. Wang, Single machine scheduling with exponential time-dependent learning effect and past-sequence-dependent setup times, *Computers & Mathematics with Applications* 57 (2009) 9–16.
- [8] J.-B. Wang, Single-machine scheduling with a sum-of-actual-processing-time based learning effect, *Journal of the Operational Research Society* 61 (2010) 172–177.
- [9] D.-L. Yang, W.-H. Kuo, Single-machine scheduling with an actual time-dependent learning effect, *Journal of the Operational Research Society* 58 (2007) 1348–1353.
- [10] R.L. Graham, E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, Optimization and approximation in deterministic sequencing and scheduling: a survey, *Annals of Discrete Mathematics* 5 (1979) 287–326.
- [11] W. Townsend, The single machine problem with quadratic penalty function of completion times: a branch-and-bound solution, *Management Science* 24 (1978) 530–534.